HAUSDORFF DIMENSION ESTIMATES FOR SOME RESTRICTED FAMILIES OF PROJECTIONS IN $\mathbb{R}^3$

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ABSTRACT. This paper is concerned with the families of projections in $\mathbb{R}^3$ onto

(i) the lines foliating the surface of a vertical cone, and
(ii) the planes perpendicular to these lines.

In case (i), I prove that if $B \subset \mathbb{R}^3$ is an analytic set with Hausdorff dimension $\dim B = s > 1/2$, then almost all projections of B have Hausdorff dimension at least $\sigma(s) > 1/2$. In case (ii) the result is similar, with the threshold $1/2$ replaced by 1.

CONTENTS

1. INTRODUCTION

This paper continues a line of research motivated by the question: are there Marstrand-Mattila type projection theorems for restricted families of projections? The original result of J. Marstrand [9] and P. Mattila [11] states that if $B \subset \mathbb{R}^d$ is an analytic set with Hausdorff dimension $\dim B \leq m$, then $\dim \pi_V(B) = \dim B$ for almost all m-dimensional subspaces $V \in G(d,m)$. Here $\pi_V \colon \mathbb{R}^d \to V$ is the orthogonal projection onto V.

In the 'restricted projections' framework, one chooses a smooth submanifold $G \subset G(d,m)$ with $\dim G < \dim G(d,m)$ and asks whether $\dim \pi_V(B) = \dim B$ for almost all $V \in G$. To date, several answers are known. First, I should mention the results of E. Järvenpää, M. Järvenpää, T. Keleti, M. Leikas and F. Ledrappier contained in the papers [7] and [6] (the latter of which generalises the theorems in the former). These papers provide a complete answer in the setting where no

2010 *Mathematics Subject Classification.* 28A80 (Primary); 28A78 (Secondary).

The research was partially supported by the Academy of Finland, grant 133264 "Stochastic and harmonic analysis, interactions and applications".

'curvature conditions' are placed on G. Indeed, [6, Theorem 3.2] gives an almost sure lower bound for $\dim \pi_V(B)$ in terms of $\dim B$ and $\dim G$. In the typical situation, there exists a number $0 < \sigma < \dim B$, depending on $\dim B$ and $\dim G$ such that $\dim \pi_V(B) \in [\sigma, \dim B]$ for almost every $V \in G$. Examples in [6] show that the lower bounds are sharp.

A natural follow-up question, whether $V \mapsto \dim \pi_V(B)$ is almost surely a constant (depending on B and G), was studied by K. Fässler and the author in [3]; positive answers were obtained in some special cases, in particular for the one-dimensional family of planes in $\mathbb{R}^3$ containing the z-axis. On the other hand, there are some trivial counterexamples, such as the concatenation of the one-dimensional families of planes in $\mathbb{R}^3$ containing the z-axis and the x-axis.

There is one notable example of a strict submanifold $G \subset G(d,m)$, for which it is known that $\dim \pi_V(B) = \dim B$ for almost all $V \in G$, and for all analytic sets B with $\dim B \leq m$. This manifold is the *isotropic Grassmannian* $G = G_h(d,m)$, a submanifold of $G(2d,m)$ with positive codimension. The projection theorem for $G_h(d,m)$ is due to Z. Balogh, K. Fässler, P. Mattila and J. Tyson [1]; a different proof based on the notion of *transversality* was given by R. Hovila [5].

As I already mentioned, the papers [7] and [6] do not impose any 'curvature conditions' on the manifold G. Concretely, this means that the setting in these papers allows for the possibility, where all the subspaces in G are contained in a single (non-trivial) subspace $W \subset \mathbb{R}^d$. Then $\pi_V(W^{\perp}) = \{0\}$, for all $V \in G$, which means that there is no dimension conservation result – in the strongest sense – for the projection family $(\pi_V)_{V \in G}$.

In a recent work [4] with K. Fässler, we proposed a curvature condition for one-dimensional submanifolds $G \subset G(3,1)$, which excludes the possibility of all (or even positively many) of the lines $L \in G$ being contained in a single plane. If the family of lines $G \subset G(3,1)$ is parametrized by a smooth curve $\gamma \colon (0,1) \to S^2$ – meaning that $G = \{\mathrm{span}(\gamma(\theta)) : \theta \in (0,1)\}$ – the condition in [4] reads as

$$\mathrm{span}\{\gamma(\theta), \dot{\gamma}(\theta), \ddot{\gamma}(\theta)\} = \mathbb{R}^3, \qquad \theta \in (0,1). \tag{1.1}$$

For such manifolds G, classical techniques, dating back as far as Kaufman's work [8] in 1968, can be used to show that the lower bounds in [7] and [6] are **not** sharp. In fact, we verified the following proposition:

Proposition 1.2 (Proposition 1.4(a) in [4]). *If $B \subset \mathbb{R}^3$ is an analytic set and $G \subset G(3,1)$ satisfies the curvature condition (1.1), then*

$$\dim \pi_L(B) \geq \min\left\{\dim B, \frac{1}{2}\right\} \quad \textit{for almost every } L \in G.$$

In contrast, the bounds in [7] and [6] give no non-trivial bounds in this situation (at least in case $\dim B \leq 1$). But even if Proposition 1.2 improves on [7] and [6] under the curvature hypothesis (1.1), there is no longer reason to believe that the bound $\min\{\dim B, 1/2\}$ is sharp. In fact, the following conjecture remains open:

Conjecture 1.3. *If $B \subset \mathbb{R}^3$ is an analytic set, and $G \subset G(3,1)$ satisfies the curvature condition (1.1), then*

$$\dim \pi_L(B) = \min\{\dim B, 1\} \quad \textit{for almost every } L \in G.$$

The main results in [4] were the verification of this conjecture for self-similar sets in $\mathbb{R}^3$ without rotations, and a slight improvement over the $\min\{\dim B, 1/2\}$ bound for *packing dimension* $\dim_{\mathrm{p}}$.

Theorem 1.4 (Theorem 1.6(a) in [4]). *If $B \subset \mathbb{R}^3$ is an analytic set with $s = \dim B > 1/2$, and G satisfies the curvature condition (1.1), then*

$$\dim_{\mathrm{p}} \pi_L(B) \geq \sigma(s) \quad \textit{for almost every } L \in G,$$

where $\sigma(s) > 1/2$ is a constant depending only on s.

The appearance of $\dim_{\mathrm{p}}$ in the theorem above was unfortunate, but the method of proof simply did not yield the same conclusion for $\dim \pi_L(B)$. The aim of the present paper is to address this issue, providing an unambiguous improvement over the bound in Proposition 1.2. However, the new ingredients in the proof make it so loaded with elementary geometry that I was no longer able to obtain the result for all families of lines satisfying (1.1). Instead, I fix one such family, namely the one consisting of lines foliating the surface of a vertical cone in $\mathbb{R}^3$. The lines are parametrized by the curve $\gamma \colon [0, 2\pi) \to S^2$, defined by

$$\gamma(\theta) = \tfrac{1}{\sqrt{2}}(\cos\theta, \sin\theta, 1),$$

and it is easy to check that γ satisfies (1.1). Writing

$$\ell_\theta := \operatorname{span}(\gamma(\theta)) \in G(3,1) \quad \text{and} \quad V_\theta := \ell_\theta^\perp \in G(3,2),$$

the main result of the paper reads as follows.

Theorem 1.5. *Let $B \subset \mathbb{R}^3$ be an analytic set, and write $s := \dim_{\mathrm{H}} B$.*

(a) *If $s > 1/2$, there exists a number $\sigma_1 = \sigma_1(s) > 1/2$ such that $\dim \pi_{\ell_\theta}(B) \geq \sigma_1$ for almost every $\theta \in [0, 2\pi)$.*
(b) *If $s > 1$, there exists a number $\sigma_2 = \sigma_2(s) > 1$ such that $\dim \pi_{V_\theta}(B) \geq \sigma_2$ for almost every $\theta \in [0, 2\pi)$.*

It is easy to pinpoint, exactly, the reason accounting for the decrease in generality from Theorem 1.4 to Theorem 1.5. It is the Three cones lemma 3.11, proved in the appendix. This lemma describes the intersection of the δ-neighbourhoods of three δ^c-separated copies of the conical surfaces

$$C = C_\gamma = \bigcup\{\operatorname{span}(\gamma(\theta)) : \theta \in [0, 2\pi)\}.$$

Apart from this lemma, the argument used in the proof of Theorem 1.5 would work equally well for all curves satisfying (1.1). However, there is no reason to believe that the analogue of Lemma 3.11 would be valid, as stated, for all cones C_γ – assuming only that (1.1) holds for γ.

The introduction is closed with a word on notation.

Notation 1.6. In order to avoid double subindices, we write

$$\rho_\theta := \pi_{\ell_\theta} \quad \text{and} \quad \pi_\theta := \pi_{V_\theta}, \qquad \theta \in [0, 2\pi).$$

Moreover, we make the identifications $\ell_\theta \cong \mathbb{R}$ and $V_\theta \cong \mathbb{R}^2$, which, concretely, means that ρ_θ maps $\mathbb{R}^3$ to $\mathbb{R}$ and π_θ maps $\mathbb{R}^3$ to $\mathbb{R}^2$.

Throughout the paper we will write $a \lesssim b$, if $a \leq Cb$ for some constant $C \geq 1$. The two-sided inequality $a \lesssim b \lesssim a$, meaning $a \leq C_1 b \leq C_2 a$, is abbreviated to $a \sim b$. Should we wish to emphasise that the implicit constants depend on a parameter p, we will write $a \lesssim_p b$ and $a \sim_p b$. The closed ball in $\mathbb{R}^d$ with centre x and radius $r > 0$ will be denoted by $B(x, r)$.

2. Acknowledgements

I am grateful to Katrin Fässler for many discussions and comments on the manuscript. The paper would not exist without our earlier work on the subject.

3. Projections onto planes

The proofs of (a) and (b) of Theorem 1.5 look very much alike, but (b) is a bit simpler technically. So I start there.

Proof of Theorem 1.5 (b). Let $B \subset \mathbb{R}^3$ be an analytic set with $\dim B = s > 1$. Make a counter assumption: there exists a compact set $E \subset [0, 2\pi)$ with positive length such that

$$\mathcal{H}^\sigma(\pi_\theta(B)) \leq 1 \quad \text{for all } \theta \in E. \tag{3.1}$$

The parameter $\sigma \in (1, s)$ will be fixed during the proof; in the end, it will only depend on s how close σ has to be chosen to one. Roughly speaking, the plan is to extract structural information about B based on our counter assumption – and to show that if σ is close to one, no s-dimensional set can have such structure.

The first task is to find small 'bad' scales $\delta > 0$, where the counter assumption (3.1) has a tractable geometric interpretation. This part of the argument is essentially due to Bourgain, see [2, p. 222].

Lemma 3.2. *Let $A \subset \mathbb{R}^d$ be a set with $\mathcal{H}^\sigma(A) \leq 1$. Then, for any $\delta_0 > 0$, there exist collections of balls $\mathcal{G}_k$, $2^{-k} < \delta_0$, with the properties that* (i) *the balls in $\mathcal{G}_k$ have bounded overlap,* (ii) *they have diameter 2^{-k},* (iii) *there are no more than $\lesssim_d 2^{k\sigma}$ balls in $\mathcal{G}_k$, and*

$$A \subset \bigcup_{2^{-k} < \delta_0} \bigcup_{B \in \mathcal{G}_k} B. \tag{iv}$$

Proof. By the very definition of $\mathcal{H}^\sigma(A) \leq 1$, one may find collections of balls $\tilde{\mathcal{G}}_k$, $2^{-k} < \delta_0$, satisfying conditions (ii)–(iv). In order to have (i), one may need to modify the collections $\tilde{\mathcal{G}}_k$ slightly. Here is how to proceed: fix k and consider the auxiliary collection of balls

$$\mathcal{G}_k' := \Big\{ B(x, 2^{-k-1}) : x \in \bigcup_{B \in \tilde{\mathcal{G}}_k} B \Big\}.$$

Then, the union of the balls in $\mathcal{G}_k'$ certainly covers the union of the balls in $\tilde{\mathcal{G}}_k$, and there is a ball in $\mathcal{G}_k'$ centred at every point of this union. Hence, the Besicovitch covering theorem, see [10, Theorem 2.7], is applicable. The conclusion is that there exists a countable subcollection $\mathcal{G}_k \subset \mathcal{G}_k'$ such that the balls in $\mathcal{G}_k$ have bounded overlap, and

$$\bigcup_{B \in \tilde{\mathcal{G}}_k} B \subset \bigcup_{B \in \mathcal{G}_k} B.$$

The proof is finished, except for the cardinality estimate $\operatorname{card} \mathcal{G}_k \lesssim 2^{k\sigma}$. To obtain this, define a function $f \colon \mathcal{G}_k \to \tilde{\mathcal{G}}_k$, by setting

$$f(B(x, 2^{-k-1})) := B,$$

if $x \in B \in \tilde{\mathcal{G}}_k$. If there are multiple admissible B, choose any one of them. By the bounded overlap condition for $\mathcal{G}_k$, no ball $B \subset \mathbb{R}^d$ of diameter 2^{-k} can contain the centres of more than C_d balls $B(x, 2^{-k-1}) \in \mathcal{G}_k$ (since all such balls contain the centre of B). Hence, the mapping f is at most C_d-to-1, and one finds that $\operatorname{card} \mathcal{G}_k \leq C_d \cdot \operatorname{card} \tilde{\mathcal{G}}_k \lesssim_d 2^{k\sigma}$. □

Fix $\delta_0 > 0$ and $\theta \in E$. Based on the counter assumption (3.1) and the lemma above, find collections $\mathcal{G}_{\theta,k}$, $2^{-k} < \delta_0$, of discs in $\mathbb{R}^2$ such that the properties (i)–(iv) listed in the lemma are satisfied with $A = \pi_\theta(B) \subset \mathbb{R}^2$. Without loss of generality, assume that $B = \operatorname{spt} \mu \subset B(0,1)$, where μ is a Frostman probability measure on $\mathbb{R}^3$ satisfying

$$\mu(B(x,r)) \lesssim r^s \quad \text{for } x \in \mathbb{R}^3 \text{ and } r > 0.$$

Then, Lemma 3.2 (iv) implies that

$$\sum_{2^{-k} < \delta_0} \mu(\pi_\theta^{-1}(\cup \mathcal{G}_{k,\theta})) \geq 1,$$

where $\cup \mathcal{G}_{k,\theta}$ stands for the union of the discs in $\mathcal{G}_{k,\theta}$. In particular, there exists $k \in \mathbb{N}$ with $2^{-k} < \delta_0$ such that

$$\mu\left(\pi_\theta^{-1}\left(\cup \mathcal{G}_{k,\theta}\right)\right) \gtrsim k^{-2}. \tag{3.3}$$

Since the conclusion holds for every $\theta \in E$, one may further pigeonhole $k \in \mathbb{N}$ so that (3.3) holds for all $\theta \in E_k \subset E$, where $|E_k| \gtrsim_{|E|} k^{-2}$. For this $k \in \mathbb{N}$, write $\delta := 2^{-k} < \delta_0$, $\mathcal{G}_\theta := \mathcal{G}_{k,\theta}$ and $E_\delta := E_k$. In the sequel, whenever the text says 'by taking $\delta > 0$ small enough' or something similar, one should understand it as 'first choose $\delta_0 > 0$ small enough, and then run through the pigeonholing argument above to find $\delta < \delta_0$'.

Given $\theta \in [0, 2\pi)$ and $x, y \in \mathbb{R}^3$, define the relation $x \sim_\theta y$ by

$$x \sim_\theta y \quad \Longleftrightarrow \quad x, y \in \pi_\theta^{-1}(B) \text{ for some } B \in \mathcal{G}_\theta.$$

So, the condition $x \sim_\theta y$ means that x and y share a common 'δ-tube' in $\mathbb{R}^3$. We now define the energy $\mathcal{E}$ by

$$\mathcal{E} := \int_0^{2\pi} \mu \times \mu(\{(x,y) : x \sim_\theta y\})\, d\theta = \iint |\{\theta \in [0,2\pi) : x \sim_\theta y\}|\, d\mu x\, d\mu y.$$

The next aim is to bound $\mathcal{E}$ from below; this will be accomplished using the first expression above. Fix $\theta \in E_\delta$. Then (3.3) holds, so there is a collection of δ-tubes $T_1, \ldots, T_N$ of the form $T_j = \pi_\theta^{-1}(B_j)$, $B_j \in \mathcal{G}_\theta$, such that the total μ-mass of the tubes T_j is $\gtrsim (\log 1/\delta)^{-2}$, and $N \lesssim \delta^{-\sigma}$. For each T_j, one has $T_j \times T_j \subset \{(x,y) : x \sim_\theta y\}$. Using this fact, the bounded overlap of the product sets $T_j \times T_j$ and the Cauchy-Schwarz inequality, one obtains the following estimate:

$$\begin{aligned}\mu \times \mu(\{(x,y) : x \sim_\theta y\}) &\gtrsim \sum_{j=1}^N [\mu(T_j)]^2 \\ &\geq \frac{1}{N}\left(\sum_{j=1}^N \mu(T_j)\right)^2 \\ &\gtrsim \delta^\sigma \cdot \mu\left(\bigcup_{j=1}^N T_j\right)^2 \\ &\gtrsim \delta^\sigma \cdot \left(\log\left(\frac{1}{\delta}\right)\right)^{-4}.\end{aligned}$$

Integrating over $\theta \in E_\delta$ and recalling that $|E_\delta| \gtrsim (\log(1/\delta))^{-2}$ yields

$$\mathcal{E} \gtrsim \delta^\sigma \cdot \left(\log\left(\frac{1}{\delta}\right)\right)^{-6}. \tag{3.4}$$

The next question is: what structural information about $B = \operatorname{spt}\mu$ does (3.4) provide? Write

$$\mathcal{C} := \bigcup_{\theta \in [0,2\pi)} \overline{\ell_\theta(\delta)},$$

where $\ell_\theta := \operatorname{span}(\gamma(\theta)) = \pi_\theta^{-1}\{0\}$. Thus, $\mathcal{C}$ is the closed δ-neighbourhood of a vertical conical surface in $C \subset \mathbb{R}^3$. The rest of the proof runs as follows. If $\delta > 0$ is small, one uses (3.4) to find three points $x_1, x_2, x_3 \in \mathbb{R}^3$ such that

$$\min\{|x_i - x_j| : 1 \leq i < j \leq 3\} \geq \delta^\kappa, \tag{3.5}$$

and

$$\mu((x_1 + \mathcal{C}) \cap (x_2 + \mathcal{C}) \cap (x_3 + \mathcal{C})) \geq \delta^\kappa. \tag{3.6}$$

Here $\kappa > 0$ is a number depending on s and σ with the central property that κ can be chosen arbitrarily close to zero by letting $\sigma \searrow 1$. On the other hand, there is Lemma 3.11 below, stating that if three cones in $\mathbb{R}^3$ are relatively well separated, then the intersection of their δ-neighbourhoods is contained in the small neighbourhood of the union of two lines in $\mathbb{R}^3$. But μ is a Frostman measure

with index $s > 1$, so such neighbourhoods cannot have too much μ-mass. This will, eventually, show that (3.5) and (3.6) are mutually incompatible for small enough $\kappa > 0$.

Remark 3.7. Here, it would be enough to consider the intersection of two cones, instead of three. Namely, if $|p - q| \geq \delta^{\kappa}$, then $(p + \mathcal{C}) \cap (q + \mathcal{C})$ is contained in the small neighbourhood of a certain smooth one-dimensional manifold, and so $\mu((p + \mathcal{C}) \cap (q + \mathcal{C}))$ is small for all Frostman measures μ with index $s > 1$. However, the intersection of at least three cones plays a crucial role in the proof of Theorem 1.5 (a). Consequently, to avoid studying separately the geometry of two- and three-cone intersections, I decided to use the three-cone argument already here. But, in case one is interested in generalising Theorem 1.5 for all curves γ satisfying the curvature condition (1.1), this remark shows that the task is much easier for Theorem 1.5 (b) than (a).

The hunt for the points $x_1, x_2, x_3 \in \mathbb{R}^3$ begins. First, observe that

$$\mathcal{E} = \int \int_{y+\mathcal{C}} |\{\theta \in [0, 2\pi) : x \sim_\theta y\}| \, d\mu x \, d\mu y. \tag{3.8}$$

Indeed, if $x \notin y + \mathcal{C}$, then the distance of x to any of the lines $y + \ell_\theta$, $\theta \in [0, 2\pi)$, is greater than δ, and consequently $|\pi_\theta(x - y)| > \delta$ for all $\theta \in [0, 2\pi)$. In particular, $x \not\sim_\theta y$ for all $\theta \in [0, 2\pi)$. To estimate the integral in (3.8) further, the following universal bound is needed:

Lemma 3.9. *If $x, y \in \mathbb{R}^3$ are distinct points, then*

$$|\{\theta \in [0, 2\pi) : x \sim_\theta y\}| \lesssim \frac{\delta}{|x - y|}.$$

Proof. Observe that

$$\{\theta \in [0, 2\pi) : x \sim_\theta y\} \subset \{\theta \in [0, 2\pi) : |\pi_\theta(x - y)| \leq \delta\}.$$

The length of the set on the right hand side can be estimated by studying the smooth function $\theta \mapsto |\pi_\theta(\xi)|^2$, $\xi \in S^2$. The crucial observation is that this function can have at most second order zeros. The details can be found above [4, (3.9)]. $\square$

Now, in order to estimate the right hand side of (3.8), define

$$G := \{y \in \mathbb{R}^3 : \mu(y + \mathcal{C}) \geq \delta^{\tau}\},$$

where $\tau = \kappa/7 > 0$. Write

$$\begin{aligned}\mathcal{E} = &\int_G \int_{y+\mathcal{C}} |\{\theta \in [0, 2\pi) : x \sim_\theta y\}| \, d\mu x \, d\mu y \\ &+ \int_{\mathbb{R}^3 \setminus G} \int_{y+\mathcal{C}} |\{\theta \in [0, 2\pi) : x \sim_\theta y\}| \, d\mu x \, d\mu y.\end{aligned}$$

The terms will be referred to as I_G and $I_{\mathbb{R}^3\setminus G}$. The term I_G is estimated using the bound from Lemma 3.9, and recalling the uniform bound $\mu(B(x,r)) \lesssim r^s$, $s > 1$.

$$I_G \lesssim \delta \cdot \int_G \int \frac{1}{|x-y|} \, d\mu x \, d\mu y \lesssim \delta \cdot \mu(G). \tag{3.10}$$

In order to estimate $I_{\mathbb{R}^3\setminus G}$, write $A_j(y) := \{x \in \mathbb{R}^3 : 2^j \leq |x-y| \leq 2^{j+1}\}$. For every $j \in \mathbb{Z}$ with $\delta \leq 2^j \leq 1$, couple the bound from Lemma 3.9 with the estimate $\mu((y+\mathcal{C}) \cap A_j(y)) \lesssim \min\{\delta^\tau, 2^{js}\} \leq \delta^{\tau(1-1/s)} \cdot 2^j$, valid for $y \in \mathbb{R}^3 \setminus G$.

$$\begin{aligned} I_{\mathbb{R}^3\setminus G} &\lesssim \int_{\mathbb{R}^3\setminus G} \int_{B(y,\delta)} d\mu x \, d\mu y \\ &\quad + \int_{\mathbb{R}^3\setminus G} \sum_{\delta \leq 2^j \leq 1} \int_{(y+\mathcal{C})\cap A_j(y)} |\{\theta \in [0,2\pi) : x \sim_\theta y\}| \, d\mu x \, d\mu y \\ &\lesssim \delta^s + \delta \cdot \int_{\mathbb{R}^3\setminus G} \sum_{\delta \leq 2^j \leq 1} 2^{-j} \cdot \mu((y+\mathcal{C}) \cap A_j(y)) \, d\mu y \\ &\lesssim \delta^s + \delta^{1+\tau(1-1/s)} \cdot \log\left(\frac{1}{\delta}\right). \end{aligned}$$

Comparing the upper bounds for I_G and $I_{\mathbb{R}^3\setminus G}$ with the lower bound (3.4) results in

$$\delta^\sigma \cdot \left(\log\left(\frac{1}{\delta}\right)\right)^{-6} \lesssim \delta \cdot \mu(G) + \delta^s + \delta^{1+\tau(1-1/s)} \cdot \log\left(\frac{1}{\delta}\right).$$

One of the three terms on the right hand side must dominate the left hand side. The middle term clearly can never do that, since $\sigma < s$. Neither can the last term, if one chooses $\sigma < 1 + \tau(1-1/s)$. Then, the only possibility remaining is that

$$\mu(G) \gtrsim \delta^{\sigma-1} \geq \delta^\tau.$$

In other words, if the counter assumption is strong enough (σ is close enough to one), the 'good set' G has relatively large μ measure. This will easily yield the existence of the three points $x_1, x_2, x_3 \in \mathbb{R}^3$. First, one uses Hölder's inequality to make the following estimate:

$$\begin{aligned} A &:= \iiint \mu((x_1+\mathcal{C}) \cap (x_2+\mathcal{C}) \cap (x_3+\mathcal{C})) \, d\mu x_1 \, d\mu x_2 \, d\mu x_3 \\ &= \iiint \int \chi_{x_1+\mathcal{C}}(y)\chi_{x_2+\mathcal{C}}(y)\chi_{x_3+\mathcal{C}}(y) \, d\mu y \, d\mu x_1 \, d\mu x_2 \, d\mu x_3 \\ &= \int \mu(y-\mathcal{C})^3 \, d\mu y \geq \left(\int \mu(y-\mathcal{C}) \, d\mu y\right)^3 \geq \left(\int_G \mu(x+\mathcal{C}) \, d\mu x\right)^3 \gtrsim \delta^{6\tau}. \end{aligned}$$

Recall that the aim is to find three points $x_1, x_2, x_3 \in \operatorname{spt}\mu \subset B(0,1)$ such (3.6) holds and the mutual distance of the points x_i is at least $\delta^\kappa = \delta^{7\tau}$. If this cannot be done, then

$$\min\{|x_i - x_j| : 1 \leq i < j \leq 3\} \geq \delta^{7\tau} \implies \mu((x_1+\mathcal{C}) \cap (x_2+\mathcal{C}) \cap (x_3+\mathcal{C})) < \delta^{7\tau}$$

for all $x, y, z \in \operatorname{spt} \mu$. Thus,

$$A \leq \sum_{1 \leq i_1 \leq i_2 \leq i_3 \leq 3} \iint \int_{B(x_{i_1}, \delta^{7\tau}) \cup B(x_{i_2}, \delta^{7\tau})} d\mu x_{i_3} \, d\mu x_{i_1} \, d\mu x_{i_2}$$
$$+ \iiint_{\{\min\{|x_i - x_j| : 1 \leq i < j \leq 3\} \geq \delta^{7\tau}\}} \delta^{7\tau} \, d\mu x_1 \, d\mu x_2 \, d\mu x_3 \lesssim \delta^{7s\tau} + \delta^{7\tau}.$$

Since $s > 1$, for small enough $\delta > 0$ this violates the lower for A obtained above. The conclusion is that there exist points $x_1, x_2, x_3 \in B(0,1)$ satisfying (3.5) and (3.6). Moreover, one may assume that $x_1 = 0$.

Now, it is time to state Three cones lemma:

Lemma 3.11 (Three cones lemma)**.** *There is an absolute constant $c > 0$ such that the following holds for small enough $\delta > 0$. Let $C \subset \mathbb{R}^3$ be the cone with opening angle $90°$, centred at the origin, and let $p, q \in B(0,1)$ be points satisfying*

$$\min\{|p|, |q|, |p - q|\} \geq \delta^c.$$

Write

$$\mathcal{C}_0 := C(\delta), \quad \mathcal{C}_p := p + \mathcal{C}_0, \quad \mathcal{C}_q := q + \mathcal{C}_0.$$

Then the intersection

$$(\mathcal{C}_0 \cap \mathcal{C}_p \cap \mathcal{C}_q) \cap B(0,1)$$

is contained in the δ^c-neighbourhood of at most two of the lines on C.

The lengthy proof of Lemma 3.11 is postponed to the appendix. The proof of Theorem 1.5 (b) is completed as follows. Applying the three cones lemma with $p = x_2$ and $q = x_3$, and choosing $\kappa < c$, one infers that $(x_1 + \mathcal{C}) \cap (x_2 + \mathcal{C}) \cap (x_3 + \mathcal{C})$ is contained in the union $L_1(\delta^c) \cup L_2(\delta^c)$ of the δ^c-neighbourhoods of two lines $L_1, L_2 \subset \mathbb{R}^3$. This means that

$$\mu((x_1 + \mathcal{C}) \cap (x_2 + \mathcal{C}) \cap (x_3 + \mathcal{C})) \leq \mu(L_1(\delta^c) \cup L_2(\delta^c)) \lesssim \delta^{c(s-1)}. \tag{3.12}$$

Choosing $\kappa < c(s-1)$, this bound contradicts (3.6) and completes the proof of the theorem. □

4. Projections onto lines

On first sight, it appears that an argument of the previous kind cannot work for the projections $\rho_\theta \colon \mathbb{R}^3 \to \mathbb{R}$, since, in this situation, one is mainly interested in sets $B \subset \mathbb{R}^3$ with $\dim B < 1$. To elaborate, let μ be a Frostman measure with index $s < 1$, and (building on the counter assumption that $\dim \rho_\theta(B) \approx 1/2$ for many $\theta \in [0, 2\pi)$) assume that one manages to find three points $x_1, x_2, x_3 \in \mathbb{R}^3$ such that (3.6) holds. Then, applying the three cones lemma, one arrives at an analogue of (3.12). Only this time, there is no non-trivial estimate to be had for the μ measure of the union $L_1(\delta^c) \cup L_2(\delta^c)$, since μ can easily be supported on this union. This may actually happen: one may choose B to lie on a single line on the cone C – the union of the lines $\operatorname{span}(\gamma(\theta))$ – and then

$$\mu((x_1 + C) \cap (x_2 + C) \cap \ldots \cap (x_m + C)) = 1 \tag{4.1}$$

for any $m \in \mathbb{N}$, for any choice of points $x_1, \ldots, x_m \in B$, and for any probability measure μ supported on B.

Nonetheless, a very simple additional trick suffices to steer clear of situations like (4.1) – and to make the proof work almost exactly as above. Fixing a Frostman measure μ supported on B, one first studies the 'bad' directions $\theta \in [0, 2\pi)$ such that a large part of the mass of μ lies near a single line in direction θ. One observes that such directions θ can only be very few (roughly speaking they have zero measure), and so they can be completely neglected. By this, I mean that the counter assumption $\dim \rho_\theta(B) \approx 1/2$ must also hold for sufficiently many 'good' directions θ. Building the proof around such directions, one may complete the argument very much in the same spirit as before.

Proof of Theorem 1.5 (a). For the additional trick described above, one needs to consider the family of projections onto planes $\tilde{\pi}_\theta \colon \mathbb{R}^3 \to \tilde{V}_\theta$, $\theta \in [0, 2\pi)$, where

$$\tilde{V}_\theta = \operatorname{span}(b_\theta)^\perp,$$

and b_θ is the line $b_\theta = \operatorname{span}(\gamma(\theta) \times \dot{\gamma}(\theta)) = \operatorname{span}((\cos\theta, \sin\theta, -1))$. As before, it suffices to prove Theorem 1.5 (a) in the case $B = \operatorname{spt} \mu \subset B(0,1)$, where μ is a Borel probability measure on $\mathbb{R}^3$ satisfying

$$I_s(\mu) := \iint \frac{d\mu x \, d\mu y}{|x-y|^s} < \infty.$$

and the growth condition $\mu(B(x,r)) \lesssim r^s$ for all balls $B(x,r) \subset \mathbb{R}^3$. Moreover, one may assume that $1/2 < s < 1$. Under these hypotheses, one has

$$\int_0^{2\pi} I_s(\tilde{\pi}_{\theta\sharp}\mu) \, d\theta < \infty, \tag{4.2}$$

where $\tilde{\pi}_{\theta\sharp}\mu$ is the measure on $\tilde{V}_\theta$ defined by $\tilde{\pi}_{\theta\sharp}\mu(A) = \mu(\tilde{\pi}_\theta^{-1}(A))$. The finiteness of the integral in (4.2) follows from the sub-level set estimate

$$|\{\theta : |\tilde{\pi}_\theta(x)| \leq \lambda\}| \lesssim \lambda,$$

valid for all $x \in S^2$ and all sufficiently small $\lambda > 0$. For more details on how to prove (4.2), see [4, §3.1], in particular [4, (3.9)].

From (4.2), one sees that $|\{\theta : I_s(\tilde{\pi}_{\theta\sharp}\mu) \geq C\}| \to 0$ as $C \to \infty$. Combining this fact with a counter assumption to Theorem 1.5 (a), one finds a constant $C > 0$ and a compact positive length set $E \subset [0, 2\pi)$ with the properties that

$$I_s(\tilde{\pi}_{\theta\sharp}\mu) \leq C, \qquad \theta \in E, \tag{4.3}$$

and

$$\mathcal{H}^\sigma(\rho_\theta(B)) \leq 1, \qquad \theta \in E. \tag{4.4}$$

This time, $\sigma > 1/2$ is a parameter close to $1/2$, to be fixed in the course of the proof. The assumption (4.3) is the additional trick: it guarantees that tubes perpendicular to the planes $\tilde{V}_\theta$ cannot carry too much μ mass. This is quantified by the following lemma:

Lemma 4.5. *Let ν be a probability measure on $\mathbb{R}^2$. Then*

$$\nu(B) \leq I_s(\nu)^{1/2} d(B)^{s/2}$$

for all ν-measurable sets $B \subset \mathbb{R}^2$.

Proof. Observe that

$$\int_0^\infty \nu \times \nu(\{(x,y) : |x-y|^{-s} \geq \lambda\}) \, d\lambda = I_s(\nu). \tag{4.6}$$

Now, let $B \subset \mathbb{R}^2$ be a ν-measurable set. Then, as long as $x, y \in B$ and $\lambda \leq d(B)^{-s}$, one has $|x-y|^{-s} \geq d(B)^{-s} \geq \lambda$. This yields the lower bound

$$\int_0^\infty \nu \times \nu(\{(x,y) : |x-y|^{-s} \geq \lambda\}) \, d\lambda \geq \int_0^{d(B)^{-s}} [\nu(B)]^2 \, d\lambda = d(B)^{-s} \cdot [\nu(B)]^2.$$

A comparison with (4.6) completes the proof. □

It follows from (4.3) and the lemma, that if T is an ϵ-tube perpendicular to a plane $\tilde{V}_\theta$, $\theta \in E$, then $\mu(T) \lesssim \epsilon^{s/2}$. However, given the counter assumption (4.4), and assuming that σ is very close to $1/2$, one can extract such tubes T with mass far greater than $\epsilon^{s/2}$. This contradiction will complete the proof, eventually.

The search for these 'bad' tubes T begins much like the search for the translated cones $x + \mathcal{C}$, as seen in the proof of Theorem 1.5 (b). The first step is to fix $\delta_0 > 0$ and find a 'bad' scale $\delta < \delta_0$ as before. This process is repeated practically verbatim, so I only state the conclusion. There exists a scale $\delta < \delta_0$, a set $E_\delta \subset E$, and collections of intervals $\mathcal{G}_\theta$, $\theta \in E_\delta$, such that

(i) for every $\theta \in E_\delta$, the collection $\mathcal{G}_\theta$ consists of $\lesssim \delta^{-\sigma}$ intervals with length δ and bounded overlap,

(ii) E_δ is compact, and

$$|E_\delta| \gtrsim \left(\log\left(\frac{1}{\delta}\right)\right)^{-2},$$

(iii)

$$\mu\left(\rho_\theta^{-1}(\cup \mathcal{G}_\theta)\right) \gtrsim \left(\log\left(\frac{1}{\delta}\right)\right)^{-2} \quad \text{for } \theta \in E_\delta.$$

The relation $x \sim_\theta y$, for $x, y \in \mathbb{R}^3$, is defined analogously with the earlier notion:

$$x \sim_\theta y \quad \Longleftrightarrow \quad x, y \in \rho_\theta^{-1}(I) \text{ for some } I \in \mathcal{G}_\theta.$$

One also defines the energy $\mathcal{E}$ almost as before by

$$\mathcal{E} := \int_{E_\delta} \mu \times \mu(\{(x,y) : x \sim_\theta y\}) \, d\theta = \iint |\{\theta \in E^\delta : x \sim_\theta y\}| \, d\mu x \, d\mu y.$$

The only difference with the earlier notion is that the domain of the θ-integration is restricted to E_δ. Following the argument leading to (3.4), one obtains the familiar lower bound

$$\mathcal{E} \gtrsim \delta^{\sigma} \cdot \left(\log\left(\frac{1}{\delta}\right)\right)^{-6}. \tag{4.7}$$

In order to estimate $\mathcal{E}$ from above, I record the following universal bound:

Lemma 4.8. *If $x, y \in \mathbb{R}^3$ are distinct points, then*

$$|\{\theta \in [0, 2\pi) : x \sim_\theta y\}| \lesssim \left(\frac{\delta}{|x-y|}\right)^{1/2}$$

Proof. Observe that

$$\{\theta \in [0, 2\pi) : x \sim_\theta y\} \subset \{\theta \in [0, 2\pi) : |\rho_\theta(x-y)| \leq \delta\}.$$

The length of the set on the right hand side can be estimated by studying the function $\theta \mapsto \rho_\theta(\xi)$, $\xi \in S^2$. The key observation is that this function can have at most second order zeros. The details can be found above [4, (3.6)]. $\square$

Next, the proof deviates a little further from the one of Theorem 1.5 (b). One defines the cone

$$C^E := \bigcup_{\theta \in E_\delta} b_\theta,$$

where $b_\theta = \operatorname{span}(\gamma(\theta) \times \dot{\gamma}(\theta)) = \operatorname{span}(\cos\theta, \sin\theta, -1)$, as before. If a difference $x - y$ stays far from C^E, the universal bound in Lemma 4.8 can be improved as follows.

Lemma 4.9. *Let $0 \leq \tau < 1$, and assume that $y - x \notin C^E(\delta^\tau)$. Then*

$$|\{\theta \in E_\delta : x \sim_\theta y\}| \lesssim \delta^{1-\tau}.$$

Proof. By definition of $C^E(\delta^\tau)$, one has $d(y-x, b_\theta) > \delta^\tau$ for all $\theta \in E_\delta$. Since $b_\theta = \ker \tilde{\pi}_\theta$, this implies that $|\tilde{\pi}_\theta(y-x)| > \delta^\tau$ for $\theta \in E_\delta$. Rewriting the inequality,

$$\left[\left((x-y)\cdot\frac{\gamma(\theta)}{|\gamma(\theta)|}\right)^2 + \left((x-y)\cdot\frac{\dot{\gamma}(\theta)}{|\dot{\gamma}(\theta)|}\right)^2\right]^{1/2} = |\tilde{\pi}_\theta(x-y)| > \delta^\tau.$$

Since $|\gamma(\theta)|$ and $|\dot{\gamma}(\theta)|$ are both bounded from below on $[0, 2\pi)$, one may infer that, for some suitable constant $c > 0$,

$$\{\theta \in E_\delta : x \sim_\theta y\} \subset \{\theta : |(x-y)\cdot\gamma(\theta)| > c\delta^\tau\} \cup \{\theta : |(x-y)\cdot\dot{\gamma}(\theta)| > c\delta^\tau\}.$$

On the other hand, the condition $x \sim_\theta y$ always implies that $|(x-y)\cdot\gamma(\theta)| \leq \delta$, so, if $\delta > 0$ is small,

$$\{\theta \in E_\delta : x \sim_\theta y\} \subset \{\theta \in [0, 2\pi) : |(x-y)\cdot\gamma(\theta)| \leq \delta \text{ and } |(x-y)\cdot\dot{\gamma}(\theta)| > c\delta^\tau\}.$$

As long as $x \neq y$, the mapping $\theta \mapsto (x-y)\cdot\gamma(\theta) = \rho_\theta(x-y)$ has at most two zeroes on $[0, 2\pi)$, and the set $\{\theta : |\rho_\theta(x-y)| \leq \delta\}$ is contained in the union of certain intervals around these zeroes. The upper bound on $|(x-y)\cdot\gamma(\theta)|$ and the

lower bound on $|(x-y)\cdot\dot\gamma(\theta)|$ show that these individual intervals have length $\lesssim \delta^{1-\tau}$, and the proof of the lemma is complete. $\square$

The next goal is to find three points $x_1, x_2, x_3 \in B(0,1)$ such that $|x_i - x_j| \geq \delta^{13\kappa}$ for $1 \leq i < j \leq 3$ and

$$\mu([x_1 + C^E(\delta^\tau)] \cap [x_2 + C^E(\delta^\tau)] \cap [x_3 + C^E(\delta^\tau)]) \geq \delta^{13\kappa}. \tag{4.10}$$

As long as one is not interested in optimising the constants in Theorem 1.5, the number τ can be chosen freely on the open interval $(0, 1/2)$; the value of $\kappa > 0$ will be fixed later, and it will have to be small relative to τ. To reach (4.10), one – almost as before – defines the set G by

$$G := \{y \in \mathbb{R}^3 : \mu(y + C^E(\delta^\tau)) \geq \delta^\kappa\}.$$

Write $\mathcal{E} = I_G + I_{\mathbb{R}^3 \setminus G}$, where

$$I_G = \int_G \int |\{\theta \in E^\delta : x \sim_\theta y\}| \, d\mu x \, d\mu y$$

and

$$I_{\mathbb{R}^3\setminus G} = \int_{\mathbb{R}^3 \setminus G} \int |\{\theta \in E^\delta : x \sim_\theta y\}| \, d\mu x \, d\mu y.$$

The part I_G is estimated using the universal bound from Lemma 4.8:

$$I_G \lesssim \delta^{1/2} \cdot \int_G \int \frac{1}{|x-y|^{1/2}} \, d\mu x \, d\mu y \lesssim \delta^{1/2} \cdot \mu(G).$$

In the latter inequality one needs the growth condition $\mu(B(x,r)) \lesssim r^s$ with some $s > 1/2$. To find an upper bound for $I_{\mathbb{R}^3\setminus G}$, another splitting of the integration is required:

$$I_{\mathbb{R}^3\setminus G} = \int_{\mathbb{R}^3\setminus G} \int_{y + C^E(\delta^\tau)} \ldots \, d\mu x \, d\mu y + \int_{\mathbb{R}^3 \setminus G} \int_{\mathbb{R}^3 \setminus (y + C^E(\delta^\tau))} \ldots \, d\mu x \, d\mu y.$$

These terms will be called $I^1_{\mathbb{R}^3\setminus G}$ and $I^2_{\mathbb{R}^3\setminus G}$. As regards $I^1_{\mathbb{R}^3\setminus G}$, the definition of $y \in \mathbb{R}^3 \setminus G$ means that $\mu(y + C^E(\delta^\tau)) < \delta^\kappa$. Let

$$A_j(y) := \{x \in \mathbb{R}^3 : 2^j \leq |x-y| \leq 2^{j+1}\}.$$

Combining the universal bound from Lemma 4.8 with the inequality

$$\mu([y + C^E(\delta^\tau)] \cap A_j(y)) \lesssim \min\{\delta^\kappa, 2^{js}\} \leq \delta^{\kappa(1-1/2s)} \cdot 2^{j/2}, \quad y \in \mathbb{R}^3 \setminus G,$$

gives

$$\begin{aligned}
I^1_{\mathbb{R}^3\setminus G} &\lesssim \int_{\mathbb{R}^3\setminus G}\int_{B(y,\delta)} d\mu x\, d\mu y\\
&\quad + \int_{\mathbb{R}^3\setminus G}\sum_{\delta\leq 2^j\leq 1}\int_{(y+C^E(\delta^\tau))\cap A_j(y)} |\{\theta\in E^\delta : x\sim_\theta y\}|\, d\mu x\, d\mu y\\
&\lesssim \delta^s + \delta^{1/2}\cdot\int_{\mathbb{R}^3\setminus G}\sum_{\delta\leq 2^j\leq 1} 2^{-j/2}\cdot\mu([y+C^E(\delta^\tau)]\cap A_j(y))\, d\mu y\\
&\lesssim \delta^s + \delta^{1/2+\kappa(1-1/2s)}\cdot\log\left(\frac{1}{\delta}\right).
\end{aligned}$$

In estimating $I^2_{\mathbb{R}^3\setminus G}$, one only needs to know that $y-x\notin C^E(\delta^\tau)$ in the inner integration. This enables the use of Lemma 4.9:

$$I^2_{\mathbb{R}^3\setminus G}\lesssim\int_{\mathbb{R}^3}\int_{\mathbb{R}^3\setminus(y+C^E(\delta^\tau))}\delta^{1-\tau}\, d\mu x\, d\mu y\leq\delta^{1-\tau}.$$

Collecting the three-part upper estimate for $\mathcal{E}$ and comparing it with the lower bound (4.7) yields

$$\delta^\sigma\cdot\left(\log\left(\frac{1}{\delta}\right)\right)^{-6}\lesssim\mathcal{E}\lesssim\delta^{1/2}\cdot\mu(G)+\delta^s+\delta^{1/2+\kappa(1-1/2s)}\cdot\log\left(\frac{1}{\delta}\right)+\delta^{1-\tau}.$$

Now, as long as $0<\kappa,\tau<1/2$ are fixed parameters, assuming that σ is close enough to $1/2$ – as one may – shows that the sum of the three last terms on the right hand side cannot dominate the left hand side for small δ. Thus, one obtains

$$\mu(G)\gtrsim\delta^{\sigma-1/2}\geq\delta^\kappa, \tag{4.11}$$

where the second inequality is, once again, reached simply by taking σ close to $1/2$. Next, as in the the proof of Theorem 1.5 (b), Hölder's inequality gives

$$A:=\iiint\mu([x+C^E(\delta^\tau)]\cap[y+C^E(\delta^\tau)]\cap[z+C^E(\delta^\tau)])\, d\mu x\, d\mu y\, d\mu z\gtrsim\delta^{6\kappa}.$$

Recall that the aim is to find a triple $x_1,x_2,x_3\in\operatorname{spt}\mu\subset B(0,1)$ such (4.10) holds and the mutual distance of the points x_i is at least $\delta^{13\kappa}$. If this cannot be done, then the condition

$$\min\{|x_i-x_j| : 1\leq i<j\leq 3\}\geq\delta^{13\kappa}$$

implies that

$$\mu([(x_1+C^E(\delta^\tau)]\cap[x_2+C^E(\delta^\tau)]\cap[x_3+C^E(\delta^\tau)])<\delta^{13\kappa}$$

for all $x_1, x_2, x_3 \in \operatorname{spt} \mu$. Thus, one finds that

$$A \leq \sum_{1 \leq i_1 \leq i_2 \leq i_3 \leq 3} \iint \int_{B(x_{i_1}, \delta^{13\kappa}) \cup B(x_{i_2}, \delta^{13\kappa})} d\mu x_{i_3} \, d\mu x_{i_1} \, d\mu x_{i_2}$$
$$+ \iiint_{\{\min\{|x_i - x_j| : 1 \leq i < j \leq 3\} \geq \delta^{13\kappa}\}} \delta^{13\kappa} \, d\mu x_1 \, d\mu x_2 \, d\mu x_3 \lesssim \delta^{13\kappa s} + \delta^{13\kappa}.$$

Since $s > 1/2$, for small enough $\delta > 0$ this violates the lower for A obtained above. Thus, there must exist points $x_1, x_2, x_3 \in B(0,1)$ such that $|x_i - x_j| \geq \delta^{13\kappa}$ and (4.10) holds. Without loss of generality, assume that $x_1 = 0$.

Assuming that $13\kappa/\tau < c$ and applying the three cones lemma with with δ^τ in place of δ, one finds that the intersection

$$(x_1 + C^E(\delta^\tau)) \cap (x_2 + C^E(\delta^\tau)) \cap (x_3 + C^E(\delta^\tau)) \cap B(0,1)$$

is contained in the $\delta^{c\tau}$-neighbourhood of at most two lines on C. Let $L_1, L_2 \subset C$ be these lines. It follows from (4.10) that either $\mu(C^E(\delta^\tau) \cap L_1(\delta^{c\tau})) \gtrsim \delta^{13\kappa}$ or $\mu(C^E(\delta^\tau) \cap L_2(\delta^{c\tau})) \gtrsim \delta^{13\kappa}$; assume that the former options holds. Then also

$$\mu(C^E(\delta^{c\tau}) \cap L_1(\delta^{c\tau})) \gtrsim \delta^{13\kappa}, \tag{4.12}$$

by monotonicity. There are two options: either L_1 forms a large angle with all the lines on $b_\theta \subset C^E$, or L_1 forms a small angle with a certain line on C^E. More precisely, assume first that the angle between L_1 and each line $b_\theta \subset C^E$, $\theta \in E$, is at least $\delta^{c\tau/2}$. Then, since L_1 intersects all the lines on C^E at the origin, simple geometry (as in [12, (4)]) shows that

$$C^E(\delta^{c\tau}) \cap L_1(\delta^{c\tau}) \subset B(0, \delta^{c\tau/3})$$

for $\delta > 0$ small enough. However, this would imply that $\mu(C^E(\delta^{c\tau}) \cap L_1(\delta^{c\tau})) \lesssim \delta^{cs\tau/3}$, which, using (4.12), can be ruled out by choosing $\kappa > 0$ small enough to begin with. The conclusion is that there exists a line $L = b_\theta \subset C^E$ such that the angle between L_1 and L is smaller than $\delta^{c\tau/2}$. It follows that $L_1(\delta^{c\tau}) \subset L(\delta^{c\tau/3})$ for small enough $\delta > 0$, and so (4.12) yields

$$\mu(L(\delta^{c\tau/3})) \gtrsim \delta^{13\kappa}.$$

To complete the proof of the theorem, apply Lemma 4.5 to the projected measure $\tilde{\pi}_{\theta\sharp}\mu$, where $L = b_\theta$. Since $\theta \in E$, one has (4.3), and then Lemma 4.5 yields an upper bound for the μ mass of the pre-images of discs on $\tilde{V}_\theta$. The neighbourhood $L(\delta^{c\tau/3})$ is such a pre-image, so

$$\mu(L(\delta^{c\tau/3})) \lesssim (\delta^{c\tau/3})^{s/2} \sim \delta^{cs\tau/6}.$$

Choosing $\kappa < cs\tau/78$, this contradicts the lower bound from (4.12) and completes the proof of Theorem 1.5 (a). $\square$

Appendix A. Proof of the Three cones lemma

Recall the statement:

Lemma A.1 (Three cones lemma)**.** *There is an absolute constant $c > 0$ such that the following holds for small enough $\delta > 0$. Let $C \subset \mathbb{R}^3$ be the cone with opening angle $90°$, centred at the origin, and let $p, q \in B(0,1)$ be points satisfying*

$$\min\{|p|, |q|, |p-q|\} \geq \delta^c.$$

Write

$$\mathcal{C}_0 := C(\delta), \quad \mathcal{C}_p := p + \mathcal{C}_0, \quad \mathcal{C}_q := q + \mathcal{C}_0.$$

Then the intersection

$$(\mathcal{C}_0 \cap \mathcal{C}_p \cap \mathcal{C}_q) \cap B(0,1)$$

is contained in the δ^c-neighbourhood of at most two of the lines on C.

The proof divides into several propositions. In order to avoid writing $'B(0,1)'$ all the time, the agreement is made that the **all** the sets below will be intersected with $B(0,1)$. Thus, any claim concerning, say, $\mathcal{C}_0 \cap \mathcal{C}_p$ should be interpreted as a claim concerning $\mathcal{C}_0 \cap \mathcal{C}_p \cap B(0,1)$ instead. A similar remark concerns the words *taking $c, \delta > 0$ small enough*: the reader should feel free to insert these words anywhere in the text, where they appear needed but missing.

Proposition A.2. *Suppose that either p or q, say p, lies in the $\delta^{1/4}$-neighbourhood of C. Then $\mathcal{C}_0 \cap \mathcal{C}_p$ (and in particular $\mathcal{C}_0 \cap \mathcal{C}_p \cap \mathcal{C}_q$) is contained in the δ^c-neighbourhood of a single line on C.*

Proof. Assume, without loss of generality, that p lies in the $\delta^{1/4}$-neighbourhood of the line $\mathrm{span}(0,1,1) \subset C$. Then $p = (0,r,r) + e$, where $|e| \leq \delta^{1/4}$ and $|r| \gtrsim \delta^c$. The idea is to study separately all the intersections $\mathcal{C}_0 \cap \mathcal{C}_p \cap H_t$, $t \in \mathbb{R}$, where H_t is the horizontal plane $H_t = \{(x,y,t) : (x,y) \in \mathbb{R}^2\} \subset \mathbb{R}^3$. Fix $t \in \mathbb{R}$ and make, temporarily, the identification $H_t \cong \mathbb{R}^2$ (more precisely: drop off the third component from all vectors on H_t). Then $\mathcal{C}_0 \cap H_t$ and $\mathcal{C}_p \cap H_t$ are contained in the δ-neighbourhoods of the circles

$$S_0 = S((0,0), |t|) \subset \mathbb{R}^2 \quad \text{and} \quad S_p = S((p_1,p_2), |p_3 - t|) \subset \mathbb{R}^2.$$

Since

$$S((p_1,p_2), |p_3 - t|) = S((e_1, r + e_2, |r + e_3 - t|),$$

where $|(e_1,e_2,e_3)| \leq \delta^{1/4}$, one may infer that the δ-neighbourhood S_p^δ is contained in the $R\delta^{1/4}$-neighbourhood of the circle $S((0,r), |r-t|)$ for some large enough absolute constant $R \geq 1$. Now, the key observation is that the circles $S(0,|t|)$ and $S((0,r), |r-t|)$ are tangent (either internally or externally) at $(0,t)$, so the intersection of their $R\delta^{1/4}$-neighbourhoods is contained in a small disc D centred at $(0,t)$. The diameter of D depends, of course, on the size of r, but choosing $c, \delta > 0$ small enough and assuming $|r| \gtrsim \delta^c$ guarantees that $\mathrm{diam}(D) \leq \delta^c$. For more details, see the proof of [12, Lemma 3.1].

Finally, observe that $(0,t,t)$ – the midpoint of D lifted from $\mathbb{R}^2$ to H_t – lies on the line $L = \mathrm{span}(0,1,1) \subset C$. Working this argument through for every $t \in \mathbb{R}$ shows that $\mathcal{C}_0 \cap \mathcal{C}_p$ is contained in the δ^c-neighbourhood of L. □

Proposition A.3. *Let A and B be sets in a metric space (X,d), and let $r,s>0$. Then*

$$A(r) \cap B(s) \subset [A(r+s) \cap B](s).$$

Proof. Let $x \in A(r) \cap B(s)$. Choose $a \in A$, $b \in B$ such that $d(x,a) \le r$, $d(x,b) \le s$. Then $b \in A(r+s) \cap B$, so that $x \in [A(r+s)\cap B](s)$. □

Proposition A.4. *There is an absolute constant $R \ge 1$ such that the intersections $\mathcal{C}_0 \cap \mathcal{C}_p$ and $\mathcal{C}_0 \cap \mathcal{C}_q$ are contained in the $R\delta^{1-c}$-neighbourhoods of the planes*

$$V_p := \left\{(x,y,z) : \left((x,y,z) - \frac{(p_1,p_2,p_3)}{2}\right) \cdot (p_1,p_2,-p_3) = 0\right\}$$

and

$$V_q := \left\{(x,y,z) : \left((x,y,z) - \frac{(q_1,q_2,q_3)}{2}\right) \cdot (q_1,q_2,-q_3) = 0\right\}.$$

Proof. By the previous proposition, it suffices to prove the claim for the intersection $C \cap \mathcal{C}_p$. Note that

$$\mathcal{C}_p = \bigcup_{r \in B(0,\delta)} p + r + C.$$

We will now prove that $C \cap (p+r+C)$ is contained in the $R\delta^{1-c}$-neighbourhood of V_p for every $r \in B(0,\delta)$. Using the equation $C = \{(x,y,z) : x^2+y^2 = z^2\}$, one easily checks that $C \cap (p+r+C)$ is contained in the plane

$$\left\{\left((x,y,z) - \frac{(p_1,p_2,p_3)+(r_1,r_2,r_3)}{2}\right) \cdot [(p_1,p_2,-p_3)+(r_1,r_2,-r_3)] = 0\right\}.$$

Now, if $(x,y,z) \in B(0,1)$ satisfies the equation above, then it follows from $|r| \le \delta$ and $p \in B(0,1)$ that

$$\left|\left((x,y,z) - \frac{(p_1,p_2,p_3)}{2}\right) \cdot (p_1,p_2,-p_3)\right| \le 3\delta.$$

Choose $(x',y',z') \in V_p$ such that the difference $(x,y,z)-(x',y',z')$ is parallel to $(p_1,p_2,-p_3)$. Then

$$\begin{aligned}
|(x,y,z)-(x',y',z')||p| &= |(x,y,z)-(x',y',z')||(p_1,p_2,-p_3)| \\
&= |[(x,y,z)-(x',y',z')] \cdot (p_1,p_2,-p_3)| \\
&= \left|(x,y,z)\cdot(p_1,p_2,-p_3) - \frac{p_1^2+p_2^2-p_3^2}{2}\right| \le 3\delta,
\end{aligned}$$

proving that (x,y,z) lies in the $(3\delta/|p|)$-neighbourhood of V_p. Since $|p| \ge \delta^c$ by hypothesis, the claim follows. □

For the remainder of the proof, fix $\tau \in (1/2,1)$.

Proposition A.5. *Assume that* $p, q \notin C(\delta^{1/4})$ *and* $\operatorname{dist}(p, \operatorname{span}(q)) \le \delta^{\tau}$. *Then the intersection* $V_p(R\delta^{1-c}) \cap V_q(R\delta^{1-c})$ *is empty. In particular, the previous lemma implies that*

$$\mathcal{C}_0 \cap \mathcal{C}_p \cap \mathcal{C}_q = \emptyset.$$

Proof. It suffices to show that the planes V_p and V_q intersected with $B(0,1)$ are at distance more than $3R\delta^{1-c}$ apart. Let $\xi = q/|q| \in S^2$, and write $p = r\xi + e$, where $|e| \le \delta^{\tau}$, and $|r - |q|| \gtrsim \delta^{1/4}$ (for the latter inequality one uses the assumption $|p - q| \ge \delta^c$ with $c \le 1/4$). Then the equation for the plane V_p becomes

$$\left\{(x, y, z) \cdot (r\xi_1 + e_1, r\xi_2 + e_2, -r\xi_3 - e_3) = \frac{(r\xi_1 + e_1)^2 + (r\xi_2 + e_2)^2 - (r\xi_3 + e_3)^2}{2}\right\}.$$

This means that if $(x, y, z) \in V_p$, then

$$(x, y, z) \cdot (\xi_1, \xi_2, -\xi_3) = r \cdot \frac{\xi_1^2 + \xi_2^2 - \xi_3^2}{2} \pm O(\delta^{\tau}) = r \cdot \frac{1 - 2\xi_3^2}{2} \pm O(\delta^{\tau}).$$

On the other hand, if $(x', y', z') \in V_q$, then

$$(x', y', z') \cdot (\xi_1, \xi_2, -\xi_3) = |q| \cdot \frac{1 - 2\xi_3^2}{2}.$$

Thus, for $(x, y, z) \in V_p$ and $(x', y', z') \in V_q$, one finds that

$$|[(x, y, z) - (x', y', z')] \cdot (\xi_1, \xi_2, -\xi_3)| \ge |r - |q|| \cdot \frac{1 - 2\xi_3^2}{2} - O(\delta^{\tau}).$$

The assumption $q \notin C(\delta^{1/4})$ shows that $\operatorname{dist}(\xi, C) \ge \delta^{1/4}$. Observing that $C \cap S^2 \subset \{(\xi_1, \xi_2, \xi_3) : \xi_3 \in \{-1/\sqrt{2}, 1/\sqrt{2}\}\}$, this implies further that

$$\operatorname{dist}(\xi_3, \{-1/\sqrt{2}, 1/\sqrt{2}\}) \gtrsim \delta^{1/4}.$$

Since the derivative of the mapping $t \mapsto 1 - 2t^2$ stays bounded away from zero near $t = \pm 1/\sqrt{2}$, one may infer that $|(1 - 2\xi_3^2)/2| \gtrsim \delta^{1/4}$. All in all, for small enough $\delta > 0$,

$$|(x, y, z) - (x', y', z')| \ge |[(x, y, z) - (x', y', z')] \cdot (\xi_1, \xi_2, -\xi_3)| \gtrsim \delta^{1/2}.$$

Since $c \le 1/4$, the term on the right hand side dominates $3R\delta^{1-c}$ for small enough $\delta > 0$. This proves that $\operatorname{dist}(V_p \cap B(0,1), V_q \cap B(0,1)) \ge 3R\delta^{1-c}$, and so the two $R\delta^{1-c}$-neighbourhoods cannot intersect inside $B(0,1)$. □

Proposition A.6. *Assume that* $\operatorname{dist}(p, \operatorname{span}(q)) \ge \delta^{\tau}$. *Then, for small enough* $c, \delta > 0$, *the intersection* $V_p(R\delta^{1-c}) \cap V_q(R\delta^{1-c})$ *is contained in the* δ^c*-neighbourhood of the the line* $V_p \cap V_q$.

Proof. Translating if necessary, one may assume that the line $L = V_p \cap V_q$ passes through the origin. Let $y \in V_p(R\delta^{1-c}) \cap V_q(R\delta^{1-c})$. Then $y = l + x$, where $l \in L$ and $x \in L^{\perp} = \operatorname{span}\{\bar{p}, \bar{q}\}$. Here $\bar{p} = (p_1, p_2, -p_3)/|p|$ and $\bar{q} = (q_1, q_2, -q_3)/|q|$ are

normal to V_p and V_q, respectively. Then, since the $\{\bar{p}, (\bar{q} - (\bar{p} \cdot \bar{q})\bar{p})/|\bar{q} - (\bar{p} \cdot \bar{q})\bar{p}|\}$ is a orthonormal basis for $\mathrm{span}\{\bar{p}, \bar{q}\}$, one sees that

$$|x| \sim |x \cdot \bar{p}| + \left| \frac{x \cdot (\bar{q} - (\bar{p} \cdot \bar{q})\bar{p})}{|\bar{q} - (\bar{p} \cdot \bar{q})\bar{p}|} \right| \leq |x \cdot \bar{p}| + \frac{|x \cdot \bar{q}|}{|\bar{q} - (\bar{p} \cdot \bar{q})\bar{p}|} + \frac{|x \cdot \bar{p}|}{|\bar{q} - (\bar{p} \cdot \bar{q})\bar{p}|}.$$

Here $|x \cdot \bar{p}|, |x \cdot \bar{q}| \leq R\delta^{1-c}$, since, for instance, $|x \cdot \bar{p}| = \mathrm{dist}(y, V_p) \leq R\delta^{1-c}$. On the other hand $|\bar{q} - (\bar{p} \cdot \bar{q})\bar{p}| \geq \delta^{\tau}$ by assumption, so one obtains $\mathrm{dist}(y, L) = |x| \lesssim \delta^{1-c-\tau}$. Hence, the claim is true as long as $c < 1 - c - \tau$.

□

Proposition A.7. *Let L be an arbitrary line in $\mathbb{R}^3$. Then the intersection $L(\delta^c) \cap \mathcal{C}_0$ is contained in the $\delta^{c^2/5}$-neighbourhood of at most two lines on C.*

Proof. Let L be the line $L = \{r\xi + p : r \in \mathbb{R}\}$, where $\xi \in S^2$ and $p \in \mathbb{R}^3$. Assume first that ξ forms a small angle with one of the lines on C, say $d(\xi, C) \leq \delta^{c/4}$. Then, if $q \in L(\delta^c)$, one may conclude that $L(\delta^c) \subset q + C(\delta^{c/5})$ for small enough $\delta > 0$. Thus, assuming that $L(\delta^c)$ intersects $\mathcal{C}_0$ at even one point, say $q \in \mathcal{C}_0$, then $L(\delta^c) \cap \mathcal{C}_0$ is certainly contained in the intersection $C(\delta^{c/5}) \cap (q + C(\delta^{c/5}))$. But now Proposition A.2 is applicable and shows that $C(\delta^{c/5}) \cap (q + C(\delta^{c/5}))$ is contained in the $\delta^{c^2/5}$-neighbourhood of a single line on C.

Next, assume that $d(\xi, C) \geq \delta^{c/4}$. By Proposition A.3, it suffices to prove that $L(\delta^c) \cap C$ is contained in the union of two small balls centred at points on C. The neighbourhood $L(\delta^c)$ is the union of the lines $L_q := \{r\xi + q : r \in \mathbb{R}\}$, where $q \in p + B(0, \delta^c)$. We may explicitly find the (at most) two points on $L_q \cap C$, since such points must satisfy

$$(r\xi_1 + q_1)^2 + (r\xi_2 + q_2)^2 - (r\xi_3 + q_3)^2 = 0,$$

amounting to

$$r = \frac{-2(\xi_1 q_1 + \xi_2 q_2 - \xi_3 q_3) \pm \sqrt{4(\xi_1 q_1 + \xi_2 q_2 - \xi_3 q_3)^2 - 4(\xi_1^2 + \xi_2^2 - \xi_3^2)(q_1^2 + q_2^2 - q_3^2)}}{2(\xi_1^2 + \xi_2^2 - \xi_3^2)}.$$

The denominator is $\gtrsim \delta^{c/4}$, by the assumption $d(\xi, C) \geq \delta^{c/4}$. The numerator, on the other hand is $1/2$-Hölder continuous with respect to moving the point $q = (q_1, q_2, q_3)$ around. So, when q ranges in $p + B(0, \delta^c)$, the solutions $r = r(q)$ can vary only inside intervals of length $\lesssim \delta^{-c/4} \cdot \delta^{c/2} = \delta^{c/4}$. This implies that the intersection $L(\delta^c) \cap C$ is contained in two balls of radius $\lesssim \delta^{c/4}$. □

Proof of the Lemma A.1. The lemma follows by combining the propositions. If either p or q lies very close to the surface C, one is instantly done by Proposition A.2. If both points lie far from the surface, then Proposition A.5 implies that either $\mathcal{C}_0 \cap \mathcal{C}_p \cap \mathcal{C}_q$ is empty, or p does not lie close to the line spanned by q. In the latter case, the intersection $\mathcal{C}_0 \cap \mathcal{C}_p \cap \mathcal{C}_q$ is contained in the small neighbourhood of a single line in $\mathbb{R}^3$, according to Proposition A.6. Finally, by Proposition A.7, the intersection of any such neighbourhood with $\mathcal{C}_0$ is contained in the neighbourhood of at most two lines on C, as claimed. □

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